Hyperbolic 4-Manifolds with Perfect Circle-Valued Morse Functions

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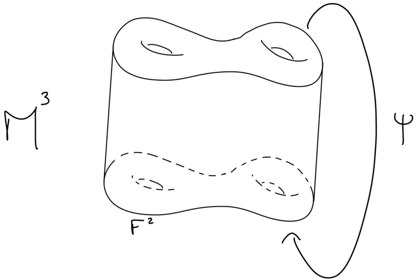

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Fibrations

A circle-valued Morse function is a fibration if it has zero critical points.

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$$\chi(M) > 0,$$

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hence we cannot have fibrations in dimension 4.

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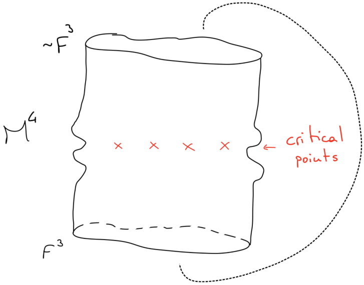

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Examples are the cyclic coverings associated with the perfect circle valued Morse functions we found.

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manifold

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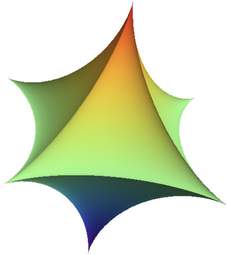
We will use a hyperbolic right-angled polytope and provide it with some structure.



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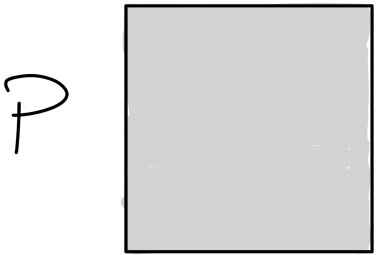
and we obtain

Hyperbolic manifold with Circle-Valued Function

(Morse and perfect)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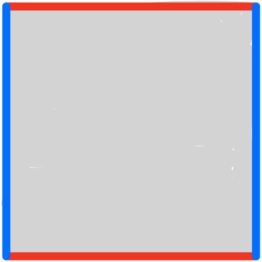

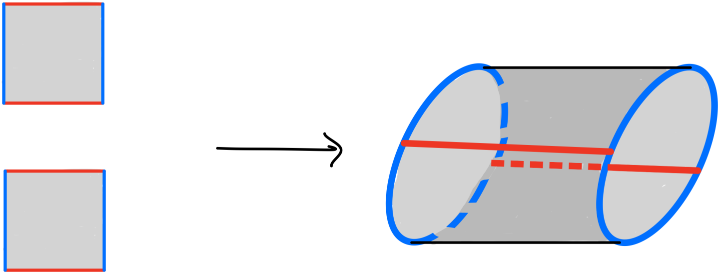

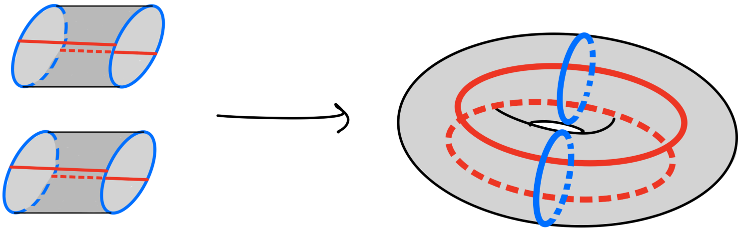

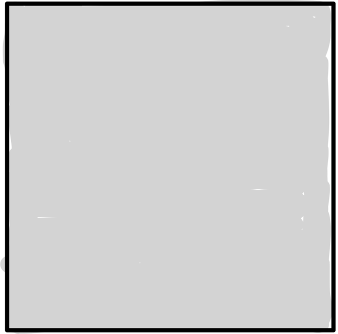

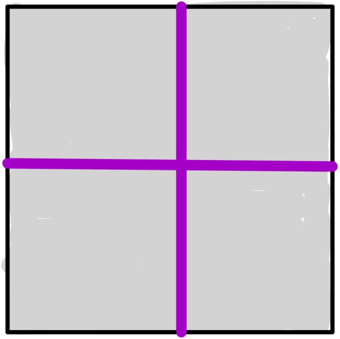

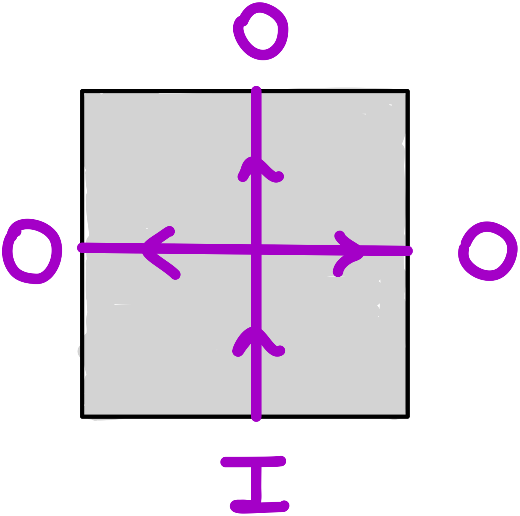
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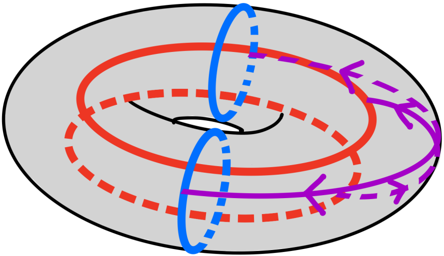

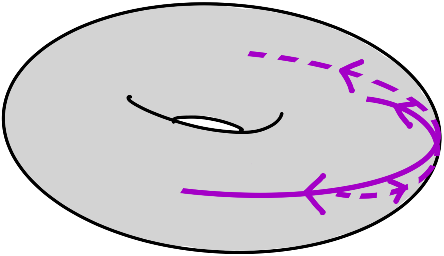

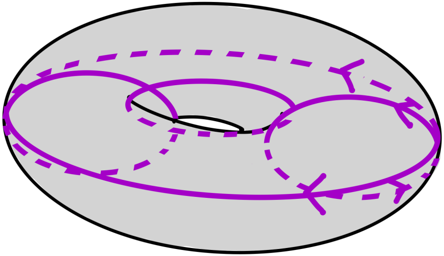

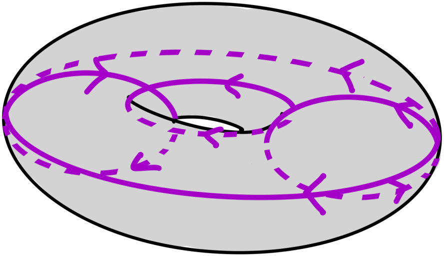

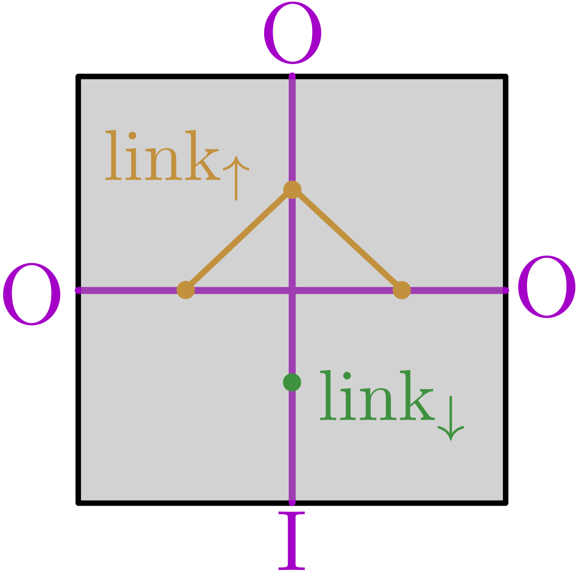
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What about the infinitesimal deformations of the cyclic coverings?

Thank you for your attention!

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