Computing deformations in hyperbolic manifolds

1 Research experience

In my academic career (consisting of my PhD completed in 2022 under the supervision of prof. Martelli and my PostDoc at University of Bologna), I focused on techniques to build and study hyperbolic manifolds, in particular using right-angled polytopes. I successfully built manifolds with specific features [5, 4] and studied their properties [3]. I have also worked on simplicial volume and bounded cohomology, writing a paper that generalizes an important result from Barge-Ghys from dimension 2 to higher dimensions [8]. I have succesfully used several computer programs and routines to obtain these results.

Then, I have held a PostDoc position at Fondazione Bruno Kessler, where I switched to an applied topic, namely Formal Methods. These consist in methodologies and computations aimed at formally proving properties of systems rather than focusing on pure mathematics. Through this experience, however, I have mastered several techniques that I have used in my mathematical work, as the usage of symbolic computations and the optimization of routines to enumerate solutions for (boolean, linear, or combinatorial) problems.

1.1 Perfect circle-valued Morse functions on hyperbolic 4-manifolds

In [5], we constructed explicit examples of both compact and cusped finite-volume hyperbolic 4-manifolds that admit **perfect circle-valued Morse functions**. Any circle-valued Morse function $f : M \to S^1$ has at least $|\chi(M)|$ critical points. We say that f is *perfect* if it has exactly $|\chi(M)|$ critical points. In odd dimension, this is equivalent to ask that f is a fibration. For hyperbolic 4-manifolds, this means that f has $\chi(M)$ critical points of index 2. Notice that, by a generalization of Gauss-Bonnet, hyperbolic manifolds in even dimension have always non-vanishing Euler characteristic. We were able to prove

Theorem 1 *There exist three explicit hyperbolic 4-manifolds (two cusped, W and X, and one closed, Z) and a hyperbolic 4-orbifold Y, each admitting perfect circle-valued Morse functions.*

Our constructions are based on coloring the facets of right-angled polytopes (*e.g.* the ideal 24-cell and the right-angled 120-cell), and applying techniques from Bestvina-Brady theory and the work of Jankiewicz-Norin-Wise to define circle-valued functions and control their singularities. The combinatorics involved in the exhibition of such examples is highly non-trivial, and we developed a computational framework (implemented in Sage, SnapPy, and Regina) to analyze the resulting manifolds, their Morse functions, and the fibers of the maps.

As a consequence, we obtained:

Theorem 2 *There exist infinitely many finite-volume hyperbolic 4-manifolds (both compact and cusped) with handle decompositions having bounded numbers of 1-handles, and thus with bounded first Betti number.*

These examples are built as finite coverings of the manifolds in Theorem 1. This result is in contrast with what happens with the second Betti number, where only a finite number of hyperbolic 4-manifolds with bounded b_2 exist [11].

In dimension three, several works are devoted to understand which homotopy classes of circle-valued functions contain a fibration. In particular, we know that this set is always the intersection of $H^1(M;\mathbb{Z})$ with a dense, polytopal set. We were able to find a similar result in one case:

Theorem 3 *For the cusped manifold W, the integral cohomology classes represented by perfect circle-valued Morse functions are the intersection of $H^1(W;\mathbb{Z})$ with a dense, polytopal subset in $H^1(W;\mathbb{R})$.*

1.2 Infinitesimal rigidity for geometrically infinite hyperbolic manifolds

In [3], I investigated the deformations of geometrically infinite hyperbolic manifolds (namely, cyclic coverings associated with perfect circle-valued Morse functions) in dimensions 4 and 5. In particular, I proved:

Theorem 4 *The infinite cyclic covering of the manifold X (described in [5], Section 2.2) associated to the map induced by the status described in [5], Section 2.2.1 and the infinite cyclic covering of the manifold M^5 associated to the map f (both described in [12], Section 1) are infinitesimally rigid.*

This work provides the first known examples of geometrically infinite hyperbolic infinitesimally rigid manifolds with finitely generated fundamental group, contrasting with the flexibility in dimension three (coming from the Density theorem for Kleinian groups). To get this result, I developed a general computational strategy to study infinitesimal rigidity of infinite cyclic coverings of cubulated manifolds, using numerical methods to find candidates and symbolic methods to prove infinitesimal rigidity. We implemented the code in Sage and MATLAB.

More specifically, we proved the infinitesimal rigidity for a finite subcomplex $F[m, n]$ of the infinite cube complex $\widetilde{C}$ corresponding to the covering. The inclusion of this finite subcomplex is shown to be π_1-surjective under mild conditions, allowing finite computations to detect rigidity. Then, we apply cohomological computations and the fundamental groupoid of the finite subcomplex to reduce the identification of the space of infinitesimal deformations to the computation of the rank of a symbolic matrix. Symbolically solving this task is far from being trivial, and several optimizations were used in order to find the exact rank in the two above-mentioned cases.

1.3 Formal methods and verification

I have explored the application of formal methods for verification of properties. In [7, 1, 2], I worked on the symbolic verification of stability and robustness properties in switched control systems, combining Lyapunov theory with SMT-based techniques. In [6], I proposed a novel framework for verifying liveness properties of hybrid systems by integrating Lyapunov-like certificates, reachability analysis, and temporal logic model checking.

While the main topics of these works are far from this proposal, I believe that the expertise acquired may help in pure mathematical tasks. As an example, in [14] the authors reduce the problem of understanding the Seiberg-Witten invariants of the Davis manifold to finding integral solutions to a big set of linear inequalities. The routine they used for this task took roughly a month of CPU time to complete, while with some optimization I was able to get the same results in less than 3 hours.

1.4 Explicit hyperbolic 4-manifold with vanishing Seiberg-Witten invariants

In [4], we manage to find the first explicit example of a hyperbolic 4-manifold with vanishing Seiberg-Witten invariants, answering a question of Agol and Lin.

Theorem 5 *There are two hyperbolic 4-manifolds N_1 and N_2 tessellated with 2^9 right-angled 120-cells with vanishing Seiberg-Witten invariants.*

The way to build such manifolds consists in finding a hyperbolic 3-manifold that is an L-space and that embeds (in a specific way) in a hyperbolic 4-manifold. To do this, we implement a procedure that makes use of Dunfield's census and the Rasmussen-Rasmussen's theorem to understand if a given manifold (given as a triangulation) is an L-space. We apply this procedure to 3-manifolds tessellated in four or less dodecahedra (call the set of these manifolds $\mathcal{D}$), proving:

Theorem 6 *Among the 29 manifolds in $\mathcal{D}$, 6 are L-spaces and 23 are not.*

We are then able to embed each of the L-spaces in corner-angled hyperbolic 4-manifolds. Using computational routines, we select the ones that have only embedded facets; hence, we can colour them to produce hyperbolic 4-manifolds with the desired property.

2 Summary of the proposed research project

There are two main lines that we propose to explore during this postdoc.

2.1 Representations of hyperbolic 3-manifold groups in $SO^o(4,1)$

Representations of fundamental groups of hyperbolic 3-manifolds in $SO^o(3,1)$ have been studied with great interest and several phenomena have been highlighted. Given an oriented 3-dimensional triangulation where the links at vertices are given by tori, the most prolific method to find a complete hyperbolic metric on the triangulation without vertices consists in associating each tetrahedon with a complex modulus $z \in \mathbb{C}$ and in finding a solution to Thurston's *consistency and completeness equations* [16]. Such a complete hyperbolic metric on the 3-manifold M is associated to the holonomy

$$\bar{\rho}\colon \pi_1(M) \to SO^o(3,1)$$

that is well defined up to conjugacy. Deformations of this representation have deep geometrical meaning: they have been used to find complete hyperbolic structures on Dehn Fillings of the manifold M; this consists in the celebrated Hyperbolic Dehn Filling Theorem. Here, we focus on deformations in a different Lie group:

Question 1 *Can we explicitely describe representations $\rho\colon \pi_1(M) \to SO^o(4,1)$ that are not conjugated to representations in $SO^o(3,1)$?*

To answer the question, we propose a generalization of Thurston's *consistency and completeness equations* from dimension 3 to dimension 4. As in dimension 3, we want to study what happens in a horosphere (now we have triangles in $\mathbb{R}^3$ instead of triangles in $\mathbb{R}^2$). We provide every tetrahedron of the triangulation with a complex modulus, and every face-gluing with a *bending parameter*. Using these quantities and fixed a triangle T in the horosphere, we can uniquely define the triangle that represents the tetrahedra glued around the corresponding face. We then -following what happens in dimension 3- write the equations that come from requiring that going around each edge e of the triangulation results in regluing the initial triangle on itself. Figure 1 gives an example of what happens when the equations are satisfied (resp. not satisfied).

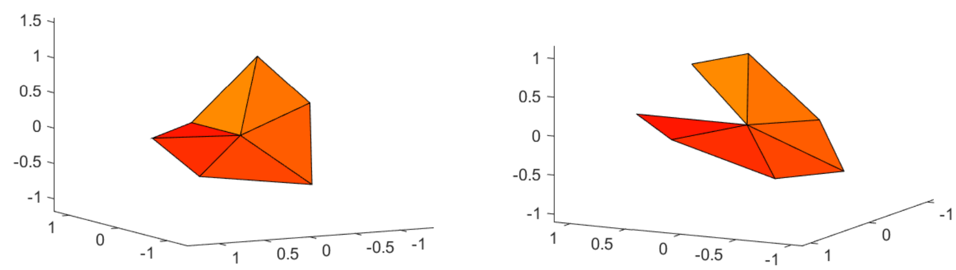


Figure 1: Behaviour of looping around an edge in a horosphere of $\mathbb{H}^4$ when gluing equations are satisfied (left) vs not satisfied (right).

These equations describe a general method to find representations of $\pi_1(M)$ in $SO^o(4,1)$. We now describe some specific lines of research that we are interested in.

2.1.1 Better understand the figure-8 example

We wrote the equations in the case of the figure-8 knot complement E. This is a well-known manifold that can be tessellated with only two tetrahedra. In this specific case, several combinatorial considerations allow to simplify the equations. For example, one can prove that all triangles must be similar to the one with sides $(1, K, K^2)$ for some $K > 1$ (the paramenter K is equal to 1 in the complete hyperbolic structure). One can also prove that the four bending parameters a, b, c, d (we have four pair of faces glued together) must be equal in pairs: $a = c$, $b = d$. To bring something concrete to the discussion, we present the equation around one edge of the triangulation:

$$\frac{A}{K^4}\, g(a,b,K) = 0$$

where

$$A = \sqrt{(K^2+K+1)(-K^2+K+1)(K^2-K+1)(K^2+K-1)}$$

is (4 times) the area of the triangle, and g is the following degree-8 polynomial in K:

$$\begin{aligned} g(a,b,K) = {} & (\cos^4 a \sin b)\, K^8 - 2(\cos^3 a \sin b)\, K^6 - 2(\sin a \cos^3 a \cos b)\, K^5 \\ & + (\sin^2 a(1+\cos^2 a) \sin b)\, K^4 + 2(\sin^3 a \cos a \cos b)\, K^3 \\ & + 2(\sin^2 a \cos^2 a \sin b)\, K^2 + 12(\sin^3 a \cos a \cos b)\, K - \sin^4 a \sin b. \end{aligned}$$

We managed to find an explicit path of solutions $\rho^E(t)$ for this equation (and the one corresponding to the other edge of the triangulation). **It can be visualized in the video present at [15].**

We plan to better understand what is happening along this path of deformations. There are several theoretical results that can be helpful in this regard. In particular, we know that any representation that sents peripheral elements to hyperbolic elements fixes a $\mathbb{H}^3$, hence it is conjugated to a representation in $SO^o(3,1)$ [13]. Therefore, the fact that we force the parabolicity of the image of peripheral elements is non-restrictive. Furthermore, by [10], we know that around $\bar{\rho}$ the variety of representations up to conjugation is contained in the union of a plane (that corresponds to deformations in $SO^o(3,1)$) and a line. We are able to explicitly show a path of representations along such line. Nevertheless, the behaviour in a neighbourhood of $\bar{\rho}$ must still be clarified. Some natural questions arise:

Question 2 *Are there discrete representations around $\bar{\rho}$? Is there a discrete representation along this path? If this is the case, does it represent anything?*

For the first question, we know that, if this is the case, then they factor through Dehn fillings:

Lemma 1 *There is a neighbourhood of $\bar{\rho}$ in the space of representation up to conjugation of $\pi_1(E)$ in $SO^o(4,1)$ where any discrete representation factors through a Dehn filling of E.*

To see this, consider that by a result of Kapovich [13], any discrete and faithful representation of $\pi_1(E)$ is conjugated to $\bar{\rho}$. Therefore, any discrete representation must be non-faithful. There is a neighborhood of $\bar{\rho}$ where hyperbolic elements are not sent to the identity, hence the kernel must consist of a parabolic element.

Using Theorem 1.4 in [10], this would strengthen Theorem 1.1 in [10] in the case of the figure-8 knot.

As far as the second question is concerned, we were actually able to identify a discrete representation of the peripheral group in this path of deformations (the final point in [15]). This is again a fruitful collaboration of numeric computations (we were able to identify a candidate representation and a candidate Dehn Filling using numerical methods) and symbolic ones (we substituted symbolic parameters in the solutions and formally verified that a relation of type $\rho(\mu^4) = \rho(\lambda)$ holds, where μ and λ are meridian and longitude of the boundary component). In particular:

Theorem 7 *There exist a non-discrete representation of $\pi_1(E)$ in $SO^o(4,1)$ that factors through a non-hyperbolic Dehn filling $E_{(4,1)}$ and whose restriction to the peripheral group is discrete.*

These kinds of results obviously need to be further investigated to try to get a general picture of what happens in the case of the figure-8 knot and (hopefully) in more generality.

2.1.2 Generalization to other 3-manifolds

We are able to write equations whose solutions provide representations of the fundamental group of a cusped 3-manifold in $SO^o(4,1)$.

Finding a symbolic solution for these equations is, however, far from straightforward.

We plan to synthesize strategies in the general setting to do this. Several geometrical arguments (as the one used in the case of the figure-8 knot complement) can be systematically used to get information on the form of a possible solution.

Furthermore, we plan to apply numerical methods to infer information on the possible space of solutions for general manifolds. This could lead to interesting insights that can then be proved theoretically. For example, the results contained in [10] apply to the complement of a 2-bridge knot, where the fundamental group is generated by two peripheral elements. One can ask:

Question 3 *What are the integrable directions of $H^1(M, \mathfrak{so}(4,1))$ when M is not the complement of a 2-bridge knot?*

This question can refer to cases where the fundamental group is generated by more peripheral elements, or where it is not generated by peripheral elements (*e.g.* manifold $m003$ in Snappy census).

Computational methods have been proven very useful in this kind of problem. For example, in [9] the authors derive properties of deformation of the holonomy of 3-manifolds inside $SL(\mathbb{R}, 4)$ using computational tools.

2.1.3 Additional directions

We here collect some additional problems that may be of interest.

Deformations of representation of the regular fiber of the perfect circle-valued Morse function. A potentially interesting application of a generic methodology to study paths of representations in $SO^o(4,1)$ comes from the work [5]. Here, one can define a representation ρ^{fiber} of the fundamental group of the regular fiber of the perfect circle-valued Morse functions inside $SO^o(4,1)$ (simply by considering the inclusion and the holonomy of the hyperbolic

4-manifold). In the cases presented in the paper, it is shown that the fiber is itself a hyperbolic 3-manifold (therefore it also admits the holonomy map $\bar{\rho}$). Hence, we have two representations of the fundamental group of the fiber inside $SO^o(4,1)$, with very different geometric properties (for example, the limit set of the first one is the entire $\mathbb{S}^3$, while the one associated with the holonomy is an $\mathbb{S}^2$). One can ask:

Question 4 *Are ρ^{fiber} and $\bar{\rho}$ connected through a path of representations? Can we explicitly describe such a path? What happens during the transition?*

Deformations of $M \times \mathbb{R}$. The composition of the holonomy $\bar{\rho}$ with the natural immersion $SO^o(3,1) \hookrightarrow SO^o(4,1)$ corresponds to a (geometrically finite, infinite volume) hyperbolic metric on $M \times \mathbb{R}$. It is unclear whether a path of deformations in $SO^o(4,1)$ has a geometrical meaning on $M \times \mathbb{R}$. One way to work on this problem on the deformations coming from our construction would be to find a triangulation of $M \times \mathbb{R}$ on which we are able to extend the deformation exhibited. This could be preliminarily done on the case of the figure-8 knot complement, and then (if a solution is found) could be generalized to other cases.

Question 5 *Is there a path of hyperbolic metrics on $E \times \mathbb{R}$ that corresponds to the path of representations $\rho^E(t)$?*

2.2 Fibers of perfect circle-valued Morse functions

2.2.1 Simplicial volume behaviour of regular fibers

The study of perfect circle-valued Morse on hyperbolic 4-manifolds is driven by a great hope. Remember the following conjecture by Whitehead:

Conjecture 1 (Whitehead asphericity conjecture - 1941) *Every connected subcomplex of a two-dimensional aspherical CW complex is aspherical.*

The simplicity of the statement, along with its year, makes it clear that any advance in such a conjecture would be of huge interest for the whole mathematical community.

The following lemma gives a possible way to tackle the conjecture:

Lemma 2 *If a cusped hyperbolic 4-manifold M admits a perfect circle-valued Morse function $f : M \to S^1$ and there exists a regular fiber M^{reg} that is not aspherical, the Whitehead asphericity conjecture is false.*

The reason is simple: the regular fiber M^{reg} is the interior of a compact 3-manifold with boundary, hence it retracts on a connected 2-dimensional CW complex. Since f is a perfect circle-valued Morse function, it lifts to a Morse function $\tilde{f}\colon \tilde{M} \to \mathbb{R}$ from the associated cyclic covering of M to $\mathbb{R}$ with only critical points of index 2. Therefore, $\tilde{M}$ can be obtained from M^{reg} by adding 2-handles, and is therefore homotopically equivalent to a 2-dimensional CW complex. Moreover, being hyperbolic, $\tilde{M}$ is aspherical.

We are unaware of topological or geometric reasons why one would expect the regular fiber of such functions to be aspherical. Nevertheless, in the cases that we have analysed, this was always the case. Driven by this motivation, we plan to systematically analyse perfect circle-valued Morse functions that arise from constructions similar to the ones presented in [5] and their singular / regular fibers. We also plan to apply the generalised techniques presented in [12] to 4-dimensional cases to increase the number of candidate functions.

In this direction, the aid of computers will be very much needed. The combinatorics involved in these processes is highly non-trivial; therefore, the usage of routines to check conditions on colorations and states is essential. The scalability of these routines on complex examples will be obtained by the usage of optimized coding languages and algorithms that ensure the correctness of the result.

2.2.2 Behaviour of regular fibers under perturbation

Perfect circle-valued Morse functions (that are fibrations in odd dimensions) can be perturbed. In particular, the homotopy type of a map $f\colon M \to \mathbb{S}^1$ corresponds to an element $[f]$ in $H^1(M, \mathbb{Z}) \subseteq H^1(M, \mathbb{R})$. It is possible to prove that adding a small element $\varepsilon \in H^1(M, \mathbb{Q})$ to $[f]$, and multiplying by the necessary constant, it is possible to find another perfect circle-valued Morse function. This shows that the homotopy classes of such functions are the intersection of an open cone in $H^1(M, \mathbb{R})$ and $H^1(M, \mathbb{Z})$. Studying how the fiber is modified during this operation is of critical interest to create and analyse more examples.

In dimension 3, it is possible to study what happens to fibers of the maps $f + \varepsilon$ when $\varepsilon \to 0$. In this case, every fibration gives a description of the manifold as $\Sigma \times [0, 1]$ where $\Sigma \times \{0\}$ is idenitified with $\Sigma \times \{1\}$ via a pseudo-Anosov map ψ. Using the description of these pseudo-Anosov maps (in particular, the existence of measured foliations), one can prove that the Euler characteristic of the fibers when ε goes to 0 goes to infinity. Hence, their simplicial volume goes to infinity.

In dimension 4, these tools are not available. In particular, the presence of index-2 critical points prevents the description of the manifold M as a mapping torus. Nevertheless, it may be possible to study what happens to some properties of the fibers when $\varepsilon \to 0$. For example, one can ask:

Question 6 *Does the simplicial volume of a regular fiber of $f + \varepsilon$ goes to infinity when ε goes to 0?*

2.3 Planned activities

The project is structured over 24 months. The first part will focus on deformations of representations, while the second one on perfect circle-valued Morse functions.

- **Months 1–3: Tool Development for representations in $SO^o(4,1)$**
  Development of computational tools for deformation equations, and acquisition of computational evidences for a census of manifolds.

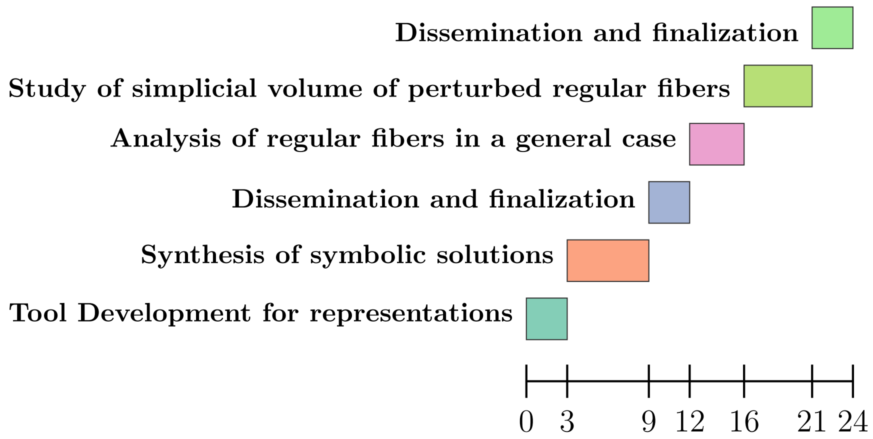
- **Months 4–9: Synthesis of symbolic solutions for other manifolds**
  Synthesis of strategies to simplify symbolically the equations in the general case and computationally synthesize the solution of the equations.

- **Months 10–12: Dissemination and finalization**
  Presentation of material to visualize the deformations in the general case, writing of papers to share the results and visiting other centers to present the work.

- **Months 13–16: Analysis of regular fibers in a general case**
  Development of tools to automatically generate perfect circle-valued Morse functions and analyse properties of their fibers.

- **Months 17–21: Study of simplicial volume of perturbed regular fibers**
  Theoretical work on properties of the perturbed fibers.

- **Months 22–24: Dissemination and finalization**
  Presentation at conferences, release of computational tools, and preparation of publications and outreach materials.


Dissemination and finalization
Study of simplicial volume of perturbed regular fibers
Analysis of regular fibers in a general case
Dissemination and finalization
Synthesis of symbolic solutions
Tool Development for representations
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